WAVES IN AN ELASTIC MEDIUM WITH COULOMB SURFACE FRICTION PRESENT

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UDC 622.011.4

We consider the propagation of waves in an elastic medium whose lateral surface is in contact with a nondeformable rough body. Slippage at the boundary may occur, giving rise to Coulomb frictional forces proportional to the normal stress in the wave. We show that when the coefficient of friction is small an equation of the following type is applicable:

$$\sigma = \sigma_0 \exp(-\delta x) \sin(\omega t - kx)$$

The absorption coefficient is a constant quantity, inversely proportional to the transverse rod dimension L, σ is the stress, ω is the circular frequency, and k is the wave number.

It is known from experiment [1] that the sound absorption coefficient in rocks is proportional to the frequency. This type of damping matches the wave damping in the medium we consider here, where the transverse dimensions are proportional to the wave length. We can, proceeding from this observation, explain our observed absorption law by assuming that in a given rock material, which ordinarily consists of granules and sheaves of various dimensions, waves of various lengths stimulate the appearance of friction on surfaces separated from one another by a distance proportional to the wave length.

A point of view different from ours was presented in [2-4] in connection with the propagation of waves in an unbounded elastic medium whose properties exhibited frictional effects.

The problem considered in [5, 6] concerned waves in an elastic rod in which the frictional forces were independent of wave amplitude. In this statement of the problem waves of sufficiently small amplitude propagate without damping.

1. Derivation of Equations

We consider a semiinfinite rod, placed along the x axis and surrounded by an incompressible medium. The equation of motion, which takes into account friction on the lateral surface of the rod, has the form

$$\rho S \frac{\partial v}{\partial t} = S \frac{\partial \sigma}{\partial x} + \tau P \tag{1.1}$$

Here ρ is the density, v is the speed, σ is the normal stress, P and S are, respectively, perimeter and area of a cross section, and τ is the tangential stress, directed along the x axis and acting on the lateral surface of the rod.

We assume that the quantity τ is proportional to the stress σ_n acting on the surface of contact and directed opposite to the speed of displacement

$$\tau = -f\,|\sigma_n|\,\mathrm{sign}\,v \tag{1.2}$$

$$v = \partial u/\partial t \tag{1.3}$$

Here f is the coefficient of friction and u is the displacement.

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 129-132, July-August, 1972. Original article submitted October 1, 1971.

For an elastic rod we obtain, upon noting that deformations in the transverse direction are zero,

$$\sigma = \rho c^2 \frac{\partial u}{\partial x}, \quad c^2 = \frac{1-\nu}{(1+\nu)(1-2\nu)} \frac{E}{\rho} \tag{1.4}$$

$$\sigma_n = -\frac{\nu}{1-\nu} \sigma \tag{1.5}$$

Here E is Young's modulus and ν is Poisson's ratio.

Eliminating the displacement from Eqs. (1.1)-(1.5), we obtain a system of two equations of hyperbolic type for the stress and the speed

$$\rho \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = -2\delta |\sigma| \operatorname{sign} v, \quad 2\delta = \frac{\nu}{1-\nu} \frac{Pf}{S} \tag{1.6}$$

$$\frac{\partial \sigma}{\partial t} - \rho c^2 \frac{\partial v}{\partial x} = 0 \tag{1.7}$$

For the displacement this system is reduced to a single second order equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -2\delta c^2 \left| \frac{\partial u}{\partial x} \right| \operatorname{sign}\left(\frac{\partial u}{\partial t} \right) \tag{1.8}$$

In the model we consider here friction appears only on the right side of the resulting equations and does not alter their hyperbolic nature. The slope of the characteristics stays constant and does not depend on the unknown quantities (if we consider a similar problem in a nonhomogeneous medium, for example, one where the sound speed depends on x, then the slope will change in accordance with changes in the properties of the medium; however, as before, it will not depend on the unknown functions).

The right side of the equations contains a nonlinear "sign" term. This nonlinearity involves a dependence on the sign of the speed and the presence of the absolute value of the stress. It is related to the introduction of the "dry" friction law in the form (1.2).

Equation (1.2) signifies the presence of the friction in both compressive and tensile phases. If we were to consider waves in a solid rod surrounded by a rigid medium, then in the tensile phase there would be a withdrawal from the wall and a corresponding absence of friction. As a result we would have a decrease in the amplitude only in the compressive phase. If, however, the rod were hollow and the incompressible medium occupied only its interior, then the amplitude decrease would be observed only in the tensile phase.

In addition to this, the compressive and tensile phases in such rods must propagate with different speeds.

An identical amplitude decrease in the compressive and tensile phases may be observed, for example, in connection with wave propagation in a thin tube in which the outer and inner perimeters are approximately equal. In this case, friction occurs at the outer surface in the compressive phase and at the inner surface in the tensile phase. Friction also occurs in both phases in a solid rod, one portion of whose lateral surface is convex and the other portion concave.

We now make some remarks on the application of Eqs. (1.4) and (1.5). We assume here that the rod deforms uniformly over its cross section. When tangential surface forces are present, this will be valid if the wave lengths λ in question are much larger than the transverse rod dimension

$$L/\lambda \ll 1, \qquad L = S/P \tag{1.9}$$

In concluding our analysis of the system of Eqs. (1.6) and (1.7), we obtain an equation for energy variation in the wave. We multiply Eq. (1.6) by v. After simple manipulations and the use of Eq. (1.7), we obtain

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \frac{\sigma^2}{2\rho c^2} \right) = \frac{\partial (\sigma v)}{\partial x} - 2\delta |\sigma v| \tag{1.10}$$

We integrate Eq. (1.10) throughout the volume of the body in which the wave is propagating. Then the first term on the right side vanishes. We obtain

$$\frac{\partial}{\partial t}\left\langle\frac{\rho v^2}{2}+\frac{\sigma^2}{2\rho c^2}\right\rangle=-2\delta\,|\sigma v| \qquad (1.11)$$

Here the angular brackets denote averaging with respect to volume. In the left member of this equation we have the change of energy in the wave per unit volume. The negative quantity in the right member is the work of the friction forces per unit time. Owing to friction the energy in the wave decreases.

2. Approximate Solution

In [7] K. E. Gubkin proposed a method for finding an approximate solution for weak shock waves of small wave length. This method is based on an approximate integration of the equations along the characteristics. We apply a similar method here for integrating Eqs. (1.6) and (1.7). As a small parameter here we take not the wave length but the coefficient of friction.

We write Eqs. (1.6) and (1.7) along the characteristics C_+ and C_-:

$$\frac{dx}{dt}=c,\quad d\sigma-\rho c\,dv=2\delta\,\mathrm{sign}(\sigma v)\,\sigma dx \qquad (2.1)$$

$$\frac{dx}{dt}=-c,\quad d\sigma+\rho c\,dv=2\delta\,\mathrm{sign}(\sigma v)\,\sigma dx \qquad (2.2)$$

We consider a wave consisting of n oscillations, each of length λ, propagating through an unperturbed homogeneous medium. The speed of the leading wave front is equal to c and the coordinate of this front is $x_1(t)$. We integrate the second relation (2.2) along the direction of the C_- characteristic, defined by the first of Eqs. (2.2)

$$\sigma+\rho cv=2\delta\int_{x_1}^{x_1-n\lambda}\mathrm{sign}(\sigma v)\,\sigma dx \qquad (2.3)$$

We estimate the magnitude of the integral according to the mean value of its integrand. When the friction is small this integral can be neglected. We obtain

$$\sigma+\rho cv=0 \qquad (2.4)$$

Here we have used the condition that the wave is propagating with respect to the unperturbed medium, i.e., $\sigma = v = 0$ for $x > x_1$.

We can estimate the applicability of Eq. (2.4) as follows:

$$2n\delta\lambda\ll 1 \quad \text{or} \quad f\ll\frac{4\pi n(1-\nu)}{\nu}\,\frac{L}{\lambda} \qquad (2.5)$$

The longer the wave train in question the smaller must be the coefficient of friction for the same degree of approximation.

Comparing the relations (2.5) and (1.9), we conclude that the frictional constraint is fairly strong. We therefore make some numerical estimates.

We take the quantity L/λ equal to 0.05. In addition we assume the condition (1.9) to be satisfied. When the Poisson ratio is taken equal to 0.3, the right side of the second of the inequalities (2.5) is equal to 1.5. Consequently, a friction coefficient of 0.1 or less will satisfy the necessary restrictions with acceptable accuracy.

Using Eq. (2.4), we integrate Eqs. (2.1) along the C_+ characteristics

$$x=ct+\alpha,\qquad \sigma=\sigma_0(\alpha)\exp(-\sigma x) \qquad (2.6)$$

Here α and $\sigma_0(\alpha)$ are quantities which are constant along a C_+ characteristic but have other values along the other C_+ characteristics. These values are determined from the initial conditions.

For example, when $x = 0$ let the stress-time dependence be given by

$$\sigma|_{x=0}=\sigma_0(t) \qquad (2.7)$$

For this case the solution of Eq. (2.6) has the form

$$\sigma=\sigma_0(t-x/c)\exp(-\delta x) \qquad (2.8)$$

All parts of the wave, regardless of their shapes, decay according to the same exponential relationship. For harmonic oscillations the damping is independent of the frequency, being determined merely by the distance traversed by the wave.

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